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Gauge Transformations and Weak Lax Equation

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Abstract

We consider several integrable systems from a standpoint of the SL(2,R) invariant gauge theory. In the Drinfeld-Sokorov gauge, we get a one parameter family of nonlinear equations from zero curvature conditions. For each value of the parameter the equation is described by weak Lax equations. It is transformed to a set of coupled equations which pass the Painlevé test and are integrable for any integer values of the parameter. Performing successive gauge transformations (the Miura transformations) on the system of equations we obtain a series of nonlinear equations.

Keywords : Integrable system, Gauge transformation, weak Lax equation

1 Introduction

Integrable systems in (1+1) dimensions have been discussed extensively so far and various techniques to analyze the integrability have been developed using the Lax pairs, the inverse scattering method, the Hirota's direct method and the Painlevé test etc.[1]. However, these methods are mainly restricted in (1+1) dimensions. In (1+1) and higher dimensions, there is a well known conjecture by Ward [2] that any integrable systems are derived from (anti) self-dual Yang-Mills gauge theories. So it is very important to re-examine integrable systems from gauge theoretical view points. The term *gauge* has been often used with several different meanings in different papers and textbooks in the

world of nonlinear mathematical physics. When we argue gauge transformations, we must clarify the status of the gauge fields or equivalently the connections. Given a gauge group, the transformation of gauge fields and the couplings with matter fields are uniquely determined.

In this paper we consider the first order formalism by Zakharov-Shabat [3] and Ablowitz-Kaup-Newell-Segur [4]. It uses a gauge covariant set of linear equations on a wave function. Since we intend to treat integrable systems in the conventional framework of gauge theory as far as possible we do not incorporate the spectral parameter unlike [4] etc. Integrable systems appear in the equation after fixing the gauge and using the zero curvature condition (ZCC)[3]. In this rather common approach, we add some new insights in this paper. They are as follows.

1) We have one parameter(α) family of nonlinear equations which only differ in gauge choice. For two specific values of α, they are well known integrable equations, *i.e.* the KdV($\alpha = 1$) and the Harry-Dym (HD)($\alpha = -2$) equations.

2) For any value of α the nonlinear equations are expressed in terms of weak Lax equations (whose spectral parameters are zero) and are transformed to ones invariant under the Möbius transformations.

3) For $\alpha = \pm$ integer, the transformed equations pass the Painlevé test and are integrable.

4) Many known and new integrable systems are produced in two ways; first by fixing the gauge with integer values of α and second by making a series of gauge transformations (the Miura transformation).

In this paper we restrict our arguments in (1+1) dimensions and the gauge group is SL(2,R) though our formulation developed in this paper may be applied to higher dimensions. Some applications of the gauge theoretical method in higher dimensions have been developed and applied to the W-symmetries [5][6].

The paper is organized as follows. In section 2 we give a one parameter family of nonlinear equations. These equations are expressed by the weak Lax equations, whose integrabilities are proved in Appendix B. Appendix C discusses their exact solutions. In section 3 we treat the KdV equation and discuss about an infinite number of conserved quantities in this framework. A series of nonlinear equations is obtained by successive gauge transformations. (We call this series as KdV sequence). The same procedures are argued on the Harry-Dym (HD) equation in section 4. Section 5 is devoted to discussions.

2 General framework

We start with a system of linear equations on a wave function covariant under some gauge group G [4]

$$D_\mu \psi \;\; = \;\; (\partial_\mu - A_\mu)\psi = 0. \tag{2.1}$$

Here A_μ is $n \times n$ matrix valued gauge field and ψ is an n column vector. It requires an

integrability condition that is nothing but the zero curvature condition (ZCC)[3],

$$F_{\mu\nu} \;\equiv\; \partial_{[\mu}A_{\nu]} \;-\; [A_\mu, A_\nu] \;=\; 0. \tag{2.2}$$

The system of equations (2.1) and (2.2) is invariant under the gauge transformation

$$A_\mu \;\rightarrow\; h\; A_\mu\; h^{-1} + \partial_\mu h\; h^{-1}, \qquad \psi \rightarrow h\; \psi, \qquad h \in G. \tag{2.3}$$

Under the infinitesimal gauge transformation of (2.3) the gauge field transforms as

$$\delta A_\mu \;=\; \partial_\mu \Lambda - [A_\mu, \Lambda], \qquad h \;\sim\; 1 + \Lambda. \tag{2.4}$$

It is convenient to write it in a similar form as the ZCC (2.2) as

$$F_{\delta\mu} \;\equiv\; \delta A_\mu - \partial_\mu \Lambda - [\Lambda, A_\mu] \;=\; 0 \tag{2.5}$$

by the following identification

$$\delta \;\leftrightarrow\; \partial_\delta, \qquad \Lambda \;\leftrightarrow\; A_\delta. \tag{2.6}$$

The general solutions have pure gauge form

$$A_\mu \;=\; \partial_\mu g\; g^{-1}, \qquad \psi = g\; \psi_0, \qquad g \;\in\; G, \qquad \psi_0 = \text{constant}, \tag{2.7}$$

and are determined up to the gauge transformation, $g \rightarrow hg$.

Suppose we fix some of the gauge freedom by imposing a set of gauge fixing conditions

$$\chi_j(A) \;=\; 0. \tag{2.8}$$

In a suitable choice of the gauge fixing conditions, the ZCC (2.2) is reduced to a set of differential equations for independent components of fields,

$$\phi_\alpha(A) = 0. \tag{2.9}$$

There remain residual transformations of (2.9) if there exist transformations preserving the equation (2.9). The existence of such residual transformations leads to an infinite set of conserved currents and is a source of the integrability of the nonlinear differential equations. It is known that various integrable systems are obtained by choosing the gauge group G and the gauge fixing conditions (2.8) depending on how the SL(2,R) subgroup is embedded in the G. We discuss the above procedure in detail for a case in (1+1) space-time dimensions and the gauge group is SL(2,R).

Parametrizing the gauge fields explicitly the equations (2.1) become

$$\partial_x \psi(t,x) \;=\; A_x(t,x)\psi, \qquad A_x \;=\; \begin{pmatrix} R & S \\ -T & -R \end{pmatrix}, \tag{2.10}$$

$$\partial_t \psi(t,x) \;=\; A_t(t,x)\psi, \qquad A_t \;=\; \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{2.11}$$

where $\psi = (\psi_1, \psi_2)^t$ and minus signature in front of T is for later convenience. The integrability condition (2.2) becomes in this case

$$\begin{aligned} R_t + cS &= a' - bT \\ S_t + 2bR - 2Sa &= b' \\ T_t + 2aT + 2cR &= -c'. \end{aligned} \tag{2.12}$$

(Here and hereafter we denote derivatives with respect to x by primes, *e.g.* b'' means $\partial_x^2 b$ etc.)

We first fix two of the three gauge freedoms as (SL(2) principal embedding)

$$R = 0 \qquad and \qquad S = 1, \tag{2.13}$$

namely

$$\partial_x \psi^{(DS)}(t,x) = \begin{pmatrix} 0 & 1 \\ -T & 0 \end{pmatrix} \psi^{(DS)}(t,x) \equiv A_x^{(DS)} \psi^{(DS)}. \tag{2.14}$$

Corresponding to this gauge fixing, the first two equations of (2.12) determine a and c of A_t

$$\partial_t \psi^{(DS)}(t,x) = \begin{pmatrix} -\frac{b'}{2} & b \\ -Tb - \frac{b''}{2} & \frac{b'}{2} \end{pmatrix} \psi^{(DS)}(t,x) \equiv A_t^{(DS)} \psi^{(DS)}. \tag{2.15}$$

We call this gauge the Drinfeld-Sokolov (DS) gauge [7]. In the DS gauge, the last equation of (2.12) is reduced to the following equation

$$T_t = 2b'T + bT' + \frac{b'''}{2}. \tag{2.16}$$

We consider infinitesimal SL(2,R) transformations consistent with the choice of A_x in (2.14). Since we have imposed two gauge fixing conditions in (2.13)@two parameters of Λ are fixed by $F_{\delta x} = 0$. There remain gauge transformations which keep the gauge fixing conditions invariant. The form of Λ is parametrized by an arbitrary function $\epsilon(t,x)$ as

$$\Lambda = \begin{pmatrix} -\frac{\epsilon'}{2} & \epsilon \\ -\epsilon T - \frac{\epsilon''}{2} & \frac{\epsilon'}{2} \end{pmatrix}. \tag{2.17}$$

The transformations of T and b are determined from $F_{\delta x} = F_{\delta t} = 0$ as

$$\delta T = 2\epsilon' T + \epsilon T' + \frac{\epsilon'''}{2}, \tag{2.18}$$

$$\delta b = \epsilon_t + \epsilon b' - b\epsilon'. \tag{2.19}$$

The remaining gauge freedom is fixed by imposing one further gauge fixing condition on T and b. In this paper we consider a case

$$T = \frac{b^\alpha}{s}, \tag{2.20}$$

where α and s are constants. By this choice the zero curvature condition (2.16) becomes

$$b_t \;=\; \frac{2+\alpha}{\alpha}\; b\, b' + \frac{s}{2\alpha}\; b^{1-\alpha}\; b'''. \tag{2.21}$$

The gauge freedoms of SL(2,R) are three and they are all fixed by choosing a specific value of α. For each α there still exist transformations (2.19) under which (2.21) remains invariant when there exist local solutions ϵ satisfying

$$\epsilon_t \;=\; \frac{2+\alpha}{\alpha}\; b\, \epsilon' + \frac{s}{2\alpha}\; b^{1-\alpha}\; \epsilon'''. \tag{2.22}$$

Every local solutions of (2.22) correspond to the residual transformations for (2.21).

We will examine the integrability of (2.21). First we can select integrable systems by the Painlevé test [8] though it is a sufficient condition but not a necessary one. One can check that only the case $\alpha = 1$ passes the Painlevé test (the value of s is irrelevant and we take $s = 2$ for convenience) . In this case $b = 2T$ and (2.21) gives the KdV equation, which will be discussed in detail in the next section.

Next we consider (2.21) in the weak Lax equation [9]. As is easily checked, (2.21) has the weak Lax pairs, L_1 and L_2,

$$\psi'' + \frac{b^\alpha}{s}\psi \equiv L_1\psi = 0, \tag{2.23}$$

$$\psi_t - b\psi' + \frac{1}{2}b'\psi \equiv L_2\psi = 0. \tag{2.24}$$

Indeed, $[L_1, L_2]\psi = 0$ gives (2.21) for arbitrary α.

Let us define φ by $\varphi \equiv \frac{\psi_1}{\psi_2}$, where ψ_i are two linearly independent solutions of (2.23) and (2.24). The concrete example of φ is discussed in Appendix A. Then φ satisfies

$$(\frac{\varphi_t}{\varphi'})^\alpha = \frac{s}{2}\{\varphi; x\}, \tag{2.25}$$

where $\{\varphi; x\}$ is the Schwarzian derivative,

$$\{\varphi; x\} \equiv \frac{\varphi'''}{\varphi'} - \frac{3}{2}\frac{\varphi''^2}{\varphi'^2}. \tag{2.26}$$

(2.25) is invariant under the Möbius transformation by construction,

$$\varphi \rightarrow \frac{a + b\varphi}{c + d\varphi}, \qquad ad - bc = 1. \tag{2.27}$$

Using the transformation of dependent variable,

$$\varphi = e^F, \tag{2.28}$$

the equation (2.25) is transformed to

$$\left(\frac{F_t}{F'}\right)^\alpha - \frac{s}{2}\left(\{F; x\} - \frac{F'^2}{2}\right) = 0. \tag{2.29}$$

Here we introduce new dependent variables [10]

$$F' \equiv u, \qquad\qquad F_t \equiv v. \tag{2.30}$$

Substituting (2.30) into (2.29), we obtain a coupled equations of u and v,

$$\left(\frac{v}{u}\right)^{\alpha} - \frac{s}{2}\left(\frac{u''}{u} - \frac{3}{2}\left(\frac{u'}{u}\right)^2 - \frac{u^2}{2}\right) = 0 \tag{2.31}$$

and

$$u_t = v'. \tag{2.32}$$

We apply the Painlevé test to the coupled equations, (2.31) and (2.32). We find, for $\alpha = \pm$ integer, that they universally have the resonances -1,1,1 and pass the Painlevé test. The Painlevé test and the exact solutions of these equations are discussed in Appendices B and C. Thus it follows (2.21) is integrable for any integer α. For $\alpha = -2, s = 1$, (2.21) gives the HD equation and (2.22) gives its residual transformations. The HD equation is a well known example which does not pass the Painlevé test in its original form. However, as we mentioned, the HD equation expressed in a coupled system (2.31) and (2.32) passes the Painlevé test and is integrable. Some related matters with the HD equation will be discussed in section 4. The equations for the other integer α are new integrable systems.

3 Gauge Transformation and the KdV Sequences

The equation corresponds to $\alpha = 1$ (and $s = 2$) in (2.20) and (2.21) gives the KdV equation,

$$T_t = T''' + 6TT'. \tag{3.1}$$

(2.25) becomes

$$\frac{f_t}{f'} = \{f; x\}, \tag{3.2}$$

where f is φ for the KdV (see Appendix A) and T is related with f by

$$T = \frac{1}{2}\{f; x\}. \tag{3.3}$$

(3.2) is called Schwarz KdV (SKdV) equation [11], [12]. (2.22) gives the residual transformations for the KdV equation,

$$\epsilon_t \;=\; \epsilon''' \;+\; 6\;T\;\epsilon'. \tag{3.4}$$

This is *the linear equation associated to the KdV equation* which Gardner et.al obtained from the inverse scattering method[13].

The equation (3.4) has an infinite number of solutions for the parameter function ϵ, which lead to an infinite number of conserved quantities of the KdV system. It is shown as follows. In writing

$$\epsilon' \;=\; \tilde{\epsilon} \tag{3.5}$$

$\tilde{\epsilon}$ satisfies, from (3.4)

$$\tilde{\epsilon}_t \;=\; \tilde{\epsilon}''' \;+\; 6\,T\,\tilde{\epsilon}' \;+\; 6\,T'\,\tilde{\epsilon}. \tag{3.6}$$

We can show δT of (2.18) is a solution of (3.6)

$$\tilde{\epsilon} \;=\; \delta T(\epsilon) \tag{3.7}$$

when T and ϵ satisfy (3.1) and (3.4) respectively. We start from a solution of (3.4), $\epsilon^{(0)} = 1$. We find a solution of (3.6) as $\tilde{\epsilon}^{(1)} = \delta T(\epsilon^{(0)}) = T'$. By integrating it we find second solution of (3.4), $\epsilon^{(1)} = T$. Repeating this procedure an infinite set of residual transformations $\epsilon^{(n)}$ of the KdV equation is obtained in local forms. They are related with an infinite number of conserved densities [14]. Wadachi showed a theorem: If ϕ_x satisfies

$$\phi_{xt} = K(\phi_x) \tag{3.8}$$

and if it has an infinite conserved density $\mathcal{G}^{(n)}$, then $\frac{\delta \mathcal{G}^{(j)}}{\delta \phi_x}$ satisfy

$$\int_{-\infty}^{\infty} dx[h(x)\frac{\partial}{\partial t}\frac{\delta \mathcal{G}^{(j)}}{\delta \phi_x} + \frac{d}{d\epsilon}K(\phi_x + \epsilon h)\frac{\delta \mathcal{G}^{(j)}}{\delta \phi_x}] = 0. \tag{3.9}$$

For the KdV equation,

$$K(\phi_x) = 6\phi_x\phi_{xx} + \phi_{4x} \tag{3.10}$$

and $\frac{\delta \mathcal{G}^{(j)}}{\delta \phi_x}$ satisfies the same equation (3.4) for ϵ. So $\epsilon^{(j)}$ are related with j-th conserved density $\mathcal{G}^{(j)}$ by

$$\frac{\delta \mathcal{G}^{(j)}}{\delta \phi_x} = \epsilon^{(j)}. \tag{3.11}$$

(3.7) is another interpretation of the bi-Hamiltonian relation [15],

$$\partial_x \frac{\delta \mathcal{G}^{(j)}}{\delta \phi_x} = (\partial_x T + T\partial_x + \frac{\partial_x^3}{2})\frac{\delta \mathcal{G}^{(j-1)}}{\delta \phi_x}. \tag{3.12}$$

Next we consider the Miura transformations from the gauge theoretical view point. Retaining $b = 2T$, we take the following gauge transformation from DS gauge. Since we have imposed three gauge fixing conditions there remains no gauge degree of freedom. Any gauge transformations (2.3) change the forms of gauge fixing conditions. Suppose we make a finite transformation (2.3) by ($h = V^{(M)}$)

$$\psi^{(M)}(t,x) = \begin{pmatrix} 1, & 0 \\ -j(t,x), & 1 \end{pmatrix} \psi^{(DS)}(t,x) \equiv V^{(M)}\psi^{(DS)}. \tag{3.13}$$

Under this transformation the gauge fields are mapped from those in the DS gauge to

$$\begin{aligned} A_x^{(M)} &= (\partial_x V^{(M)}(t,x) \;+\; V^{(M)}(t,x)A_x)V^{(M)}(t,x)^{-1} \\ &= \begin{pmatrix} j, & 1 \\ -T - j^2 - j', & -j \end{pmatrix}, \end{aligned} \tag{3.14}$$

$$
\begin{aligned}
A_t^{(M)} &= (\partial_t V^{(M)}(t,x) \ + \ V^{(M)}(t,x)A_t)V^{(M)}(t,x)^{-1} \\
&= \begin{pmatrix} -T' + 2Tj, & 2T \\ -j_t - 2T^2 - 2Tj^2 + 2jT' - T'', & T' - 2Tj \end{pmatrix}.
\end{aligned} \tag{3.15}
$$

The transformation function Λ in (2.17), which kept the gauge fixing conditions (2.13) invariant, is also mapped to

$$
\begin{aligned}
\Lambda^{(M)} &= (\delta V^{(M)} \ + \ V^{(M)}\Lambda)V^{(M)}(t,x)^{-1} \\
&= \begin{pmatrix} -\frac{\epsilon'}{2} + \epsilon j, & \epsilon \\ -\delta j - \epsilon T - \epsilon j^2 + j\epsilon' - \frac{\epsilon''}{2}, & \frac{\epsilon'}{2} - \epsilon j \end{pmatrix}.
\end{aligned} \tag{3.16}
$$

They are solutions of $F_{xt} = F_{\delta x} = F_{\delta t} = 0$ in the new gauge.

So far $j(t,x)$ in $V^{(M)}$ is not specified. We choose it as

$$
T = -j^2 - j' \tag{3.17}
$$

so that the $(A_x^{(M)})_{21}$, (2,1) component of matrix $A_x^{(M)}$, vanishes. The ZCC condition requires $(A_t^{(M)})_{21}$ to be zero also and

$$
j_t \ = \ -6j^2j' + j'''. \tag{3.18}
$$

This is the well-known MKdV equation. The compatibility requires that $(\Lambda^{(M)})_{21}$ in the gauge transformation (3.16) must vanish and

$$
\delta j \ = \ j\epsilon' + j'\epsilon - \frac{\epsilon''}{2}. \tag{3.19}
$$

This is the transformation property of T in (2.18) in terms of j. From (3.3) and (3.17)

$$
j = -\frac{f''}{f'}. \tag{3.20}
$$

We can repeat the same form of gauge transformation as (3.13) on the MKdV equation furthermore and get other integrable systems;

$$
\psi^{(C)} = \begin{pmatrix} 1 & 0 \\ -\eta(t,x) & 1 \end{pmatrix} \psi^{(M)}(t,x) \equiv V^{(C)}\psi^{(M)}, \tag{3.21}
$$

where η is chosen as $(A_x^{(C)})_{21}$ in the new gauge vanishes

$$
\eta' = -\eta^2 - 2j\eta. \tag{3.22}
$$

The ZCC determines the differential equation for η with $b = 2T$,

$$
\eta_t = \eta''' - \frac{1}{2}(3\eta'^2\eta^{-1} + \eta^3)_x. \tag{3.23}
$$

This is the Calogero Korteweg-de Vries (CKdV) equation [16]. The successive Bäcklund transformations from the KdV to CKdV equations were discussed in the bilinear formalism in [17]. The residual symmetry transformation of η is

$$
\delta\eta \ = \ (\epsilon\eta)_x. \tag{3.24}
$$

η is also related with $f(t,x)$ of the SKdV equation by

$$\eta = \frac{f'}{f}. \tag{3.25}$$

Thus by the successive gauge transformations of the type (3.13), we get a series of integrable equations, which we refer as the KdV sequence. The same series of integrable equations have been obtained by expanding the (usual) Lax equation in power series of the spectral parameter [18]. Namely we begin with

$$\psi'' + T\psi = \lambda\psi \tag{3.26}$$

and expand ψ as

$$\psi = exp(\int dx(r^{(0)} + r^{(1)}\lambda + r^{(2)}\lambda^2 + ...)). \tag{3.27}$$

Then the recursion equation of $r^{(k)}$ gives the same series of integrable equations. Thus the successive Miura transformations with no spectral parameter in our theory correspond to the expansion of the wave function by the spectral parameters in the conventional theories.

4 Harry Dym Equation

The Harry Dym (HD) equation [19],

$$b_t = -\frac{1}{4}b^3 b''', \tag{4.1}$$

corresponds to $\alpha = -2$ (and $s = 1$) in (2.21). In this gauge (2.14) and (2.15) become

$$\partial_x \psi^{(HD)}(t,x) \;=\; \begin{pmatrix} 0 & 1 \\ -\frac{1}{b^2} & 0 \end{pmatrix} \psi^{(HD)}(t,x) \equiv A_x^{(HD)}\psi^{(HD)}, \tag{4.2}$$

$$\partial_t \psi^{(HD)}(t,x) \;=\; \begin{pmatrix} -\frac{b'}{2} & b \\ -\frac{1}{b} - \frac{b''}{2} & \frac{b'}{2} \end{pmatrix} \psi^{(HD)}(t,x) \equiv A_t^{(HD)}\psi^{(HD)}. \tag{4.3}$$

The residual transformation Λ in this case is

$$\Lambda^{(HD)} \;=\; \begin{pmatrix} -\frac{\epsilon'}{2} & \epsilon \\ -\frac{\epsilon}{b^2} - \frac{\epsilon''}{2} & \frac{\epsilon'}{2} \end{pmatrix} \tag{4.4}$$

and

$$\delta b = -b\epsilon' + b'\epsilon - \frac{b^3}{4}\epsilon'''. \tag{4.5}$$

(4.1) has the residual gauge transformations, that is, (4.5) with ϵ restricted by

$$\epsilon_t = -\frac{1}{4}b^3\epsilon'''. \tag{4.6}$$

Applying (3.9) in the HD equation, we obtain the linear equation associated to the HD equation

$$\varepsilon_t \;=\; \frac{3}{4}b^2 b'''\varepsilon \;-\; \frac{1}{4}(b^3\varepsilon)''', \tag{4.7}$$

where

$$\varepsilon^{(j)} = \frac{\delta \mathcal{G}^{(j)}}{\delta b} \tag{4.8}$$

with $\mathcal{G}^{(j)}$ j-th conserved density. It should be remarked that in the HD case the associated linear equation (4.7) has different form from the equation of residual gauge transformation (4.6) unlike the case of KdV. We can check (4.7) for first several terms:

$$\begin{aligned} \mathcal{G}^{(0)} &= \frac{-1}{b}, \qquad \mathcal{G}^{(1)} = \frac{b'^2}{2b}, \qquad \mathcal{G}^{(2)} = \frac{b'^4}{8b} + \frac{bb''^2}{2}, \\ \mathcal{G}^{(3)} &= \frac{b'^6}{16b} + \frac{3bb'^2b''^2}{4} - \frac{b^2b''^3}{2} + \frac{b^3b'''^2}{2}, \ldots. \end{aligned} \tag{4.9}$$

ε in (4.7) and ϵ in (4.6) are related by

$$\epsilon = b^3 \, \varepsilon. \tag{4.10}$$

ε's satisfy the bi-Hamiltonian relations like (3.12) in the KdV case,

$$- b^2 \, \partial_x \, b^2 \, \varepsilon^{(n)} = b^3 \, \partial_x^3 \, b^3 \, \varepsilon^{(n-1)} \tag{4.11}$$

from which the conserved densities $\mathcal{G}^{(n)}$ follow. (4.10) tells that series of solutions of (4.6) is obtained from

$$- \, \partial_x \, (\frac{\epsilon^{(n)}}{b}) = b \, \partial_x^3 \, \epsilon^{(n-1)}. \tag{4.12}$$

Namely starting from $n = 0$ and $\epsilon^{(-1)} = 1$ we obtain

$$\begin{aligned} &\epsilon^{(-1)} = 1, \qquad \epsilon^{(0)} = b, \qquad \epsilon^{(1)} = b(\frac{b'^2}{2} - bb''), \\ &\epsilon^{(2)} = \frac{3\,b\,b'^4}{8} - \frac{3\,b^2\,b'^2\,b''}{2} + \frac{3\,b^3\,b''^2}{2} + 2\,b^3\,b'\,b''' + b^4\,b^{(4)}, \ldots . \end{aligned} \tag{4.13}$$

In this way we can obtain local expressions for $\epsilon^{(n)}$ and give an infinite residual symmetry of HD equation through (4.5) and (4.6).

Repeating the same procedure as in the previous section, we can get the HD sequence as follows. Retaining $T = \frac{1}{b^2}$, we perform the gauge transformations from the DS gauge and obtain, so called, the HD sequence as follows.

$$\psi^{(mHD)}(t,x) = \begin{pmatrix} 1 & 0 \\ -k(t,x) & 1 \end{pmatrix} \psi \equiv V^{(mHD)} \psi. \tag{4.14}$$

The gauge fields are transformed from (4.2) and (4.3) to

$$\begin{aligned} A_x^{(mHD)} &= (\partial_x V^{(mHD)}(t,x) \; + \; V^{(mHD)}(t,x) A_x^{HD}) V^{(mHD)}(t,x)^{-1} \\ &= \begin{pmatrix} k, & 1 \\ -k' - k^2 - \frac{1}{b^2}, & -k \end{pmatrix} \end{aligned} \tag{4.15}$$

and

$$
\begin{aligned}
A_t^{(mHD)} &= (\partial_t V^{(mHD)}(t,x) \;+\; V^{(mHD)}(t,x)A_t^{HD})V^{(mHD)}(t,x)^{-1} \\
&= \begin{pmatrix} -\frac{b'}{2}+bk, & b \\ -k_t+b'k-\frac{1}{b}-\frac{b''}{2}-bk^2, & -bk+\frac{b'}{2} \end{pmatrix}.
\end{aligned} \tag{4.16}
$$

Here we apply the same rule as in the KdV sequence also that the (2, 1) components of gauge transformed A_μ are zero:

$$
\frac{1}{b^2} = -k' - k^2, \tag{4.17}
$$

$$
-k_t + b'k - \frac{1}{b} - \frac{b''}{2} - bk^2 = 0 \tag{4.18}
$$

and

$$
\delta k \;=\; \epsilon' k + \epsilon k' - \frac{\epsilon''}{2}. \tag{4.19}
$$

In this way we can obtain a series of non-linear equations analogous to the KdV sequences, though we do not repeat the arguments furthermore. The explicit proof of integrability of this series apart from the HD equation is still open though it is formulated in the same ways as the KdV series in our formalism. Antonowicz and Fordy discussed different extensions of the HD equation in the inverse scattering framework [20]

5 Discussion

In this paper we have analyzed several integrable systems in the framework of gauge theory. Starting with the DS gauge, we have fixed the remaining gauge freedom by (2.20) and obtained the one parameter (α) family of nonlinear equations. It was shown explicitly for $\alpha = 1$ (KdV) and $\alpha = -2$ (HD) cases that the residual invariant transformations assure an infinite number of conserved quantities. The relations between the residual transformations and conserved quantities are simple for the KdV but are not for the others. The one parameter family of equations (2.21) was discussed in the framework of weak Lax equation. The set of equations (2.31) and (2.32) was shown to pass the Painlevé test for $\alpha = \pm$ integer and are integrable. Applying the gauge transformations (the Miura transformations) successively to (2.14) and (2.15) with (2.20), we have also constructed the KdV and HD sequences.

It is very interesting to discuss other typical integrable systems like the Burgers, Sawada-Kotera, Kaup-Kuperschmidt, and Sine-Gordon equations etc. in this formalism. Some of them, at least, can be incorporated in our formalism by relaxing the DS gauge. However, important is not the fact that our formalism simply invokes these known and unknown integrable systems but is that those equations are mutually related by the gauge fixings ($\alpha = \pm$ integer) and gauge transformations (the Miura transformations for each α) in the gauge theoretical framework. It is our goal to understand integrable systems in the framework of gauge theories in a self complete manners. In this paper we have partially succeeded in it but there still left many points unanswered. For instance, we

have not obtained an infinite number of conserved quantities for all integer α. Although in this paper we have restricted our arguments in (1+1) dimensions and SL(2,R) gauge group, they can be applied to higher dimensions and to other gauge groups, for example W-symmetries.

A Example of $\varphi \equiv \psi_1/\psi_2$

Let us consider the two linerly independent solutions of Eq.(2.23) for the KdV equation. Eq.(2.14) reads

$$\psi''(x,t) + T(x,t)\psi(x,t) = 0. \tag{A.1}$$

If we put

$$\psi \;=\; e^{g(x,t)}, \tag{A.2}$$

it is a solution of (A1) if T(x,t) is given by

$$T(x,t) \;=\; -\,g'(x,t)^2 - g''(x,t). \tag{A.3}$$

For another solution, we must solve

$$\psi(x,t)'' \;+\; (-g'(x,t)^2 - g''(x,t))\psi(x,t) \;=\; 0 \tag{A.4}$$

for given $g(x,t)$. It has a trivial solution $\psi = A\; e^{g(x,t)}$ for x independent A. We look for solution with x dependent A,

$$\psi \;=\; A(x,t)\; e^{g(x,t)}.$$

The equation of motion for A is obtained from Eq.(A.1).

$$A'' \;+\; 2\; g'(x,t)A' \;=\; 0, \tag{A.5}$$

which can be integrated as

$$A'(x,t) \;=\; c_1(t)\; e^{-2g(x,t)}, \quad\longrightarrow\quad A(x,t) \;=\; c_2(t) + \; c_1(t)\int^x dx'\; e^{-2g(x',t)}. \tag{A.6}$$

Thus the solution is

$$\psi(x,t) \;=\; (c_2(t) + c_1(t)\int^x dx'\; e^{-2g(x',t)})\; e^{g(x,t)}, \quad T(x,t) \;=\; -g'(x,t)^2 - g''(x,t). \tag{A.7}$$

It is the general solution containing one arbitrary function of $g(x)$ and two constants $c_1,\; c_2$ with respect to x.

The solution contains an integral expression. In place of $g(x,t)$ we can use

$$f(x,t) \;=\; \int^x dx'\; e^{-2g(x',t)} \tag{A.8}$$

as the arbitrary function describing the solution. In terms of $f(x,t)$, the general solution has the local form;

$$\psi \;=\; f'(x,t)^{-1/2}\;(c_1\;f(x,t)\;+\;c_2),\;\;T(x,t)\;=\;\frac{1}{2}\;(\;\frac{f'''(x,t)}{f'(x,t)}-\frac{3}{2}\frac{f''^2(x,t)}{f'(x,t)^2})\equiv\frac{1}{2}\{f(x,t);x\}. \tag{A.9}$$

Thus the two linearly independent solutions, for instance, are

$$f'(x,t)^{-1/2}\;(c_1\;\;f(x,t)\;\;+\;\;c_2)\equiv c_1\psi_1(x,t)+c_2\psi_2(x,t) \tag{A.10}$$

and

$$\varphi\equiv\frac{\psi_1}{\psi_2}=f(x,t) \tag{A.11}$$

The two linearly independent solutions in general are given by

$$\Psi_1=a\psi_1+b\psi_2,\quad\Psi_2=c\psi_1+d\psi_2 \tag{A.12}$$

with $ad-bc=1$ (please do not confuse the constants $a,\sim,d$ with the functions $a,\sim,d$ in Eq.(2.11)). In this case φ changes as Eq.(2.27) but Eq.(3.2) is invariant as it should be.

B Painlevé test of (2.31) and (2.32)

We apply the Painlevé test to the coupled equations, (2.31) and (2.32) for $\alpha=$ an arbitrary integer. As will been shown, the resonances are -1,1 and 1 for any integer α. First we consider the case of $\alpha=$ positive integer in which the KdV equation ($\alpha=1$) is included, and second the case of $\alpha=$ negative integer to which the HD equation ($\alpha=-2$) belongs.

B.1 Case of $\alpha=$ positive integer

For the case of $\alpha=$positive integer ($\equiv n$), (2.31) is transformed to

$$v^n-\frac{s}{2}\left(u''u^{n-1}-\frac{3}{2}u'^2u^{n-2}-\frac{u^{n+2}}{2}\right)=0. \tag{B.1}$$

As usual we expand u and v around a movable pole γ^{-1}. To obtain the leading order term, substitute

$$u=u_0\gamma^i\qquad v=v_0\gamma^j \tag{B.2}$$

with $u_0v_0\neq 0$ into (B.1) and (2.32). Then (2.32) gives $i=j$ and the dominant equation of (B.1) with this result gives i=j=-1, and

$$u_0=\pm\gamma_x\quad v_0=\pm\gamma_t\;\;\text{(double sign is in same order)} \tag{B.3}$$

irrelevantly to n. (It should be remarked that the dominant terms of (B.1) are the terms in the bracket.) So we can expand u and v as

$$u=\sum_{i=0}^{\infty}u_i\gamma^{i-1},\quad v=\sum_{j=0}^{\infty}v_j\gamma^{j-1}. \tag{B.4}$$

Substitute (B.4) into (B.1) and (2.32), perturb up to $O(\gamma^{j-n-2})$ and $O(\gamma^{j-2})$, respectively, and pick up u, v having the largest suffix j, then we get the resonance equation:

$$\begin{pmatrix} (j+1)(j-1)u_0\gamma_x^2 & 0 \\ (j-1)\gamma_t & -(j-1)\gamma_x \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} f(u_i, v_i; i \leq j) \\ u_{j-1,t} - v_{j-1,x} \end{pmatrix}. \tag{B.5}$$

Thus we have the resonances, -1,1,1.

We can check the consistency by substituting (B.4) into (B.1) and collecting the terms of order $\gamma^{-(n+1)}$, we find

$$2(n-1)u_1u_0^{n-1}\gamma_x^2 - \frac{3}{2}(n-2)u_1u_0^{n-1}\gamma_x^2 - \frac{1}{2}(n+2)u_1u_0^{n+1} \equiv 0 \tag{B.6}$$

with the help of (B.3). Thus u_1 and, therefore, v_1 from (2.32) are indefinite. The subsequent coefficients u_i, v_i $(i \geq 2)$ are definite by virtue of the first term of (B.1). Thus (2.31) and (2.32) have passed the Painlevé test.

B.2 Case of $\alpha =$ negative integer

For the case of $\alpha =$ negative integer $(\equiv -n)$, (2.31) is transformed to

$$u^n - \frac{s}{2}\left(\frac{u''}{u} - \frac{3}{2}\left(\frac{u'}{u}\right)^2 - \frac{u^2}{2}\right)v^n = 0 \tag{B.7}$$

In this case the Painlevé test is performed analogously to the positive integer case or even simpler than before since the resonance equation is determined by the factor in front of v^n in (B.7). So we do not repeat the argument.

C Exact solutions to the system of (2.31) and (2.32)

To obtain the exact solutions to the system of (2.31) and (2.32) we change the dependent variables as

$$u = a(\ln f)_x \quad v = a(\ln f)_t, \tag{C.1}$$

where we have took into consideration (2.32) and the fact that u and v have the leading order -1. We search the exact solutions for each specific value of α. First we consider the case $\alpha = 1$. Substituting (C.1) into (2.31) and setting $a = 1$ to eliminate the terms of order f^{-4}, we obtain

$$4f_tf' + 3sf''^2 - 2sf'f''' = 0 \tag{C.2}$$

This equation is same as (3.2) as is expected. This equation can not be written as the bilinear form [21] but we can obtain the dispersion relation and one soliton solution á la Hirota Method. Namely let us set $f(t,x) = 1 + \exp(px + qt)$ in (C.2), then the dispersion relation is

$$q = -\frac{s}{4}p^3. \tag{C.3}$$

Therefore it goes from (C.1) and (C.2) that

$$u = \frac{p}{2}e^{\zeta/2}\text{sech }(\zeta/2) \tag{C.4}$$

$$v = \frac{q}{2}e^{\zeta/2}\text{sech }(\zeta/2) \tag{C.5}$$

with

$$\zeta \equiv px + qt. \tag{C.6}$$

Formally we can trace back this one soliton solution to those of the KdV sequence by the relations with f like (3.3), (3.20), (3.25) etc. Unfortunately, this process gives trivial (constant) solutions for the KdV and MKdV equations. However, (3.25) gives the solution for the CKdV equation as

$$\eta = \frac{p}{2}e^{\zeta/2}\text{sech }(\zeta/2) = u. \tag{C.7}$$

(2.31) and (2.32) does not allow multi-soliton solutions. Namely let us put

$$f_2 = 1 + \epsilon(e^{p_1x+q_1t} + e^{p_2x+q_2t}) + \epsilon^2 A_{12}e^{(p_1+p_2)x+(q_1+q_2)t} \tag{C.8}$$

for two solitons solution and substitute it into (3.2). Equating coefficients of like powers of ϵ to zeros, we obtain

$$q_i = -\frac{s}{4}p_i^3 \quad (i = 1, 2) \tag{C.9}$$

for O(1),

$$-3s\exp\{-\frac{s}{4}(p_1^3 + p_2^3)t + (p_1 + p_2)x\}p_1(p_1 - p_2)^2 p_2 = 0 \tag{C.10}$$

for O(ϵ), leading to $p_1 = p_2$, and

$$12sA_{12}^2\exp(-sp_1^3t + 4p_1^4x)p_1^4 = 0 \tag{C.11}$$

for O(ϵ^4). O(ϵ^2) and O(ϵ^3) terms vanish identically. Obviously these equations are not satisfied for $p_i \neq 0$.

For the other values of α, as is easily seen from the previous arguments, the situation is not changed except for replacing the dispersion (C.3) by

$$q = (\pm)^{\alpha+1}p(-\frac{s}{4}p^2)^{1/\alpha}. \tag{C.12}$$

There is no multi solitons solution. This is a natural consequence from the fact that (2.31) and (2.32) can not be written in bilinear forms. However, as was shown in Appendix B this system of equations is integrable.

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